ON INVERSE PROBLEMS IN TURBINE ENGINE PARAMETER ESTIMATION

M. Henriksson  
Performance & Control systems  
Volvo Aero Corporation AB  
Trollhättan, Sweden  
Email: mattias.mh.henriksson@volvo.com

S. Borguet  
University of Liège  
Turboengineering Group  
Liège, Belgium  
Email: s.borguet@ulg.ac.be

O. Léonard  
University of Liège  
Turboengineering Group  
Liège, Belgium  
Email: o.leonard@ulg.ac.be

T. Grönstedt  
Chalmers University of Technology  
Department of Applied Mechanics  
Göteborg, Sweden  
Email: tomas.gronstedt@chalmers.se

ABSTRACT

This paper extends previous work on model order reduction based on singular value decomposition. It is shown how the decrease in estimator variance must be balanced against the bias on the estimate inevitably introduced by solving the inverse problem in a reduced order space. A proof for the decrease in estimator variance by means of multi-point analysis is provided. The proof relies on comparing the Cramer-Rao lower bound of the single point and the multi-point estimators. Model order selection is discussed in the presence of a varying degree of a priori parameter information, through the use of a regularization parameter. Simulation results on the SR-30 turbojet engine indicate that the theoretically attainable multi-point improvements are difficult to realize in practical jet engine applications.

Keywords: Gas turbine performance estimation, least-squares, Cramer-Rao bounds, multi-point techniques, model order reduction.

NOMENCLATURE

$E(\cdot)$  Mathematical expectation operator  
$FN$  Net thrust  
$H$  Influence Coefficient Matrix or model matrix  
$mse$  Mean squared error  
$N$  Spool speed  
$\mathcal{N}(m, C)$  the Gaussian probability density function with mean $m$ and covariance matrix $C$  
$Pi$  Total pressure at station $i$  
$SEi$  Map scaling factor for efficiency of the component located at section $i$  
$SNR$  Signal to noise ratio  
$SWiR$  Map scaling factor for flow capacity of the component whose entry is located at section $i$  
$Ti$  Total temperature at station $i$  
$WFE$  Fuel flow  
$W1$  Inlet air mass flow  
$x$  Signal vector  
$y$  Measurement vector
The shortage on sensor data typically experienced in the field of jet engine performance estimation and fault detection has led the research down two roads of remedy. The first approach is to make use of a priori information on the parameters, e.g. [4,6], and the second is to extend the analysis to a multiple operating point formulation [17]. It is of course possible to combine the two approaches.

Several authors have illustrated that some improvement in identifiability may result from the use of multiple operating point analysis [8,17]. This paper formalizes these observations by deriving a Cramer-Rao lower bound [15] on the variance of the single and multiple operating point estimators (see Appendix). However, identifiability improvements obtained by multi-point analysis are usually moderate [7], and without an appropriate model reduction technique making efficient use of the increased data set, these improvements may show to be of limited value. A simulation study is carried out in the paper to support this claim.

Previous work on gas turbine model order reduction has introduced the use of singular value decomposition to obtain an optimal reduced parameter space [9,12]. For a given model order, estimator variance is then in general lower than in any suboptimal physical parameter space. However, previous work has only shown how to perform the model reduction optimally, but has not discussed how far this process should be driven. Although the reduction process promises to reduce estimator variance, any order reduction will also inevitably introduce an estimator bias. The design of the inverse problem must then include balancing these two counteracting factors in order to minimize estimator mean square error (mse). This process is outlined as part of analyzing the estimator design for the SR-30 turbojet engine.

**STATEMENT OF THE PROBLEM**

**Linear Estimation Problem Formulation**

The parameter estimation problem, using a linear model, is to determine the parameter vector \( \theta \) that is related to given observations \( y \) according to:

\[
y = H \theta + \varepsilon
\]

where \( y \in \mathbb{R}^m \), \( \theta \in \mathbb{R}^n \) and \( H \in \mathbb{R}^{m \times n} \), \( \varepsilon \in \mathcal{N}(0,R) \).

In the field of turbine engines, the problem of parameter estimation is to determine a subset of the parameter vector \( \theta \): the performance parameters \( \theta_p \in \mathbb{R}^p \). The remaining part \( \theta_u \in \mathbb{R}^u \), so-called deviation parameters, consists of the unknown deviation in the control inputs. The single-point problem can therefore be written as:

\[
y = \begin{pmatrix} H_p & H_u \end{pmatrix} \begin{pmatrix} \theta_p \\ \theta_u \end{pmatrix} + \varepsilon
\]

The assumption in this paper, which is common to all multi-point approaches, is that the interesting parameters \( \theta_p \) can be regarded as unknown constants even at different power settings. On the contrary, the deviation parameters \( \theta_u \) may vary with the power setting and inlet conditions. In turbine engine parameter estimation, multi-point estimation is often proposed as a solution

\[
\text{Subscripts}
\]

- \( \alpha \): Regularization parameter
- \( \varepsilon \): Measurement noise
- \( \sigma \): Standard deviation
- \( \theta \): Parameter vector
- (\( \ast \)): Estimate
to enhance the accuracy, see [7, 17] or [8]. For instance, the 2-point estimation problem is:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} H_{p,1} & H_{a,1} \\ H_{p,2} & H_{a,2} \end{pmatrix} \begin{pmatrix} \theta_p \\ \theta_{a,1} \theta_{a,2} \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \tag{3}$$

**Bound on the Estimator Variance**

A formal way of assessing the performance of an estimator is to compute its Cramer-Rao lower bound [10]. Briefly explained, the Cramer-Rao bound introduces a lower limit on the variance of an unbiased estimator, effectively limiting the uncertainty in the $\theta_p$ parameter.

Although the system represented by equation (3) has an increased number of unknowns, the least-squares estimator achieves a better performance, i.e., agreement between model outputs and measured outputs, when working with multiple operating point estimation. The single point and multi-point estimators are both analyzed with respect to their Cramer-Rao lower bounds as given in the Appendix.

**Selection of the Multiple Operating Points**

The first question that has to be addressed when formulating a multi-point estimation problem is to decide on the most suitable operating points. Insight into this process can be obtained by deriving a linear process model from the nonlinear engine model at different power settings, and acquiring the covariance of the estimated parameters based on standard Gaussian assumptions on the measurement noise.

Note that the linear system model in equation (1) can be scaled to a linear system model with noise distributions as $\varepsilon \in \mathcal{N}(0, I)$ provided that the covariance $R$ is positive definite. The best linear unbiased estimate, see [10], is given by $\hat{\theta} \in \mathcal{N}((\theta_0, (H^T \hat{H}))^{-1}$, where the transformed model is $\hat{H} = (\sqrt{R})^{-1} H$.

If the a priori measurement uncertainties $\varepsilon_1$ and $\varepsilon_2$, see equation (3), are assumed to be zero-mean Gaussian distributed, $\varepsilon_1, \varepsilon_2 \in \mathcal{N}(0, I)$, then the covariance of the estimation error $R_p = [(\theta_p - \hat{\theta}_p) (\theta_p - \hat{\theta}_p)^T]$ can be derived as, see [10]:

$$R_p = (I_p 0 0) H_{MP}^T H_{MP}^{-1} (I_p 0 0) \tag{4}$$

The maximum power setting is a natural choice as one observation point because it exhibits the best signal to noise ratio. Then, a low power operating point is selected to maximize the addition of information provided by the non-linearity of the model.

Any additional operating point is then added only if the Frobenius norm of the covariance matrix $R_p$ is reduced significantly, as compared to the reduction obtained when adding the second, low power, operating point.

**LINEAR ESTIMATION METHODS**

**Weighted Regularized Least-Squares**

The weighted regularized least-squares solution is formulated by introducing covariance matrices for the measurement error and the parameters. The measurement error covariance matrix is denoted $R$ according to:

$$R = E(\varepsilon \varepsilon^T) \tag{5}$$

where $\varepsilon$ is assumed to be zero-mean, Gaussian distributed.

The covariance matrix for the parameter vector, also considered as a Gaussian random variable, is:

$$Q = E(\theta \theta^T)$$

The estimated health parameters are obtained by solving the following quadratic optimization problem:

$$\min_{\hat{\theta}} (y - H\theta)^T R^{-1} (y - H\theta) + (\theta - \theta_0)^T Q^{-1} (\theta - \theta_0) \tag{6}$$

where the first term is a quadratic penalization of the residuals and the second one restricts the estimated parameters to lie in a neighborhood of the prior value $\theta_0$.

The solution of the optimization problem (6) is:

$$\hat{\theta} = (H^T R^{-1} H + Q^{-1})^{-1} (H^T R^{-1} y + Q^{-1} \theta_0) \tag{7}$$

**A Unified Framework for Linear Least-Squares and Bayesian Estimation**

The problem of Bayesian linear estimation (6) can also be written as a classical least-squares problem by introducing the prior value on the parameters as additional measurements:

$$\begin{pmatrix} y \\ \theta_0 \end{pmatrix} = \begin{pmatrix} H \\ I_n \end{pmatrix} \theta + \begin{pmatrix} \varepsilon \\ v \end{pmatrix} \tag{8}$$

where $v$ is assumed to be zero-mean Gaussian distributed: $v \in \mathcal{N}(0, \alpha^{-1} Q)$. The parameter $\alpha$ is introduced to allow a variable weighting between the residuals and the regularization term in the estimator. Specifically, when $\alpha$ is close to zero, this corresponds to an absence of a priori knowledge on the parameters, resulting in the classical least-squares method.
Equation (8) can be interpreted as having more information available than equation (1). However, in case of a sudden change in the performance parameter vector \( \theta_p \), it is a good idea to decrease – or even remove – the \textit{a priori} information on these parameters:

\[
\begin{pmatrix} y \\ \theta_{0\, u} \end{pmatrix} = \begin{pmatrix} H_p & H_u \\ 0 & I_u \end{pmatrix} \theta + \begin{pmatrix} \epsilon \\ \nu \end{pmatrix} \tag{9}
\]

The linear estimation problem can also be formulated as:

\[
y = H_p \theta_p + H_u \theta_u + \hat{\epsilon} \tag{10}
\]

where \( \theta_u \in \mathcal{N}(0, Q_u) \) and \( \hat{\nu} \in \mathcal{N}(0, H_u Q_u H_u^T + R) \). The best linear unbiased estimate of \( \theta_p \) is given by equation (7), see [10,16].

**Reduced Order Estimator**

Reduced order modeling by means of singular value decomposition may sometimes be convenient. Model order reduction has previously been applied to the field of gas turbines, see for example [12] or [9]. The idea is that the “true” model may not be the most suited one to estimate the parameters.

![Figure 1. Comparison between full and reduced order estimation](image_url)

The full rank projection matrix is:

\[
P_H = H (H^T H)^{-1} H^T \tag{11}
\]

By replacing a full rank estimator with a reduced rank one, the variance will decrease. On the other hand a bias is introduced in the estimate. Consequently, a trade-off between variance and bias must be found by carefully selecting the order of the reduced model.

The system matrix \( H \) is replaced with a reduced rank projection matrix that can be determined from a singular value decomposition. First, apply a singular value decomposition to the system matrix \( H \):

\[
H = U S V^T \tag{12}
\]

The reduced matrix \( H_r \) is obtained by removing the \( k \) smallest singular values and corresponding column vectors of \( U \) and \( V \). The reduced projection matrix \( P_r \) is found by introducing the new \( H_r \) into the full rank expression (11).

\[
\begin{align*}
H &= (U_r \ U_n) \begin{pmatrix} S_r & 0 \\ 0 & S_n \end{pmatrix} V^T \\
H_r &= U_r (S_r, 0) V^T \\
P_r &= U_r U_r^T
\end{align*} \tag{13}
\]

where the new projection matrix has the following property:

\[
P_r P_H = P_r
\]

The solution of the estimation problem can be interpreted as a least-square solution where the measurements are projected by \( P_r \):

\[
\hat{\theta}_r = (H_r^T H_r)^{-1} H_r^T P_r, y = (H_r^T H_r)^{-1} H_r^T P_r, y \tag{14}
\]

In Figure 1, a geometrical interpretation of the true signal vector \( x = H \theta \) and the true reduced model signal vector \( x_r = H_r \theta \) can be found. Together with the estimated signal vectors \( \hat{x} = P_H y \) and \( \hat{x}_r = P_r, y \), where the projection matrices project the measurement vector, \( y \), onto the subspaces \( < U_r > \) and \( < U_H > \). Without any reduction, the signal error is unbiased:

\[
x - \hat{x} = x - P_H y \in \mathcal{N}(0, P_H)
\]

On the contrary, with a reduced order model, a bias is introduced as:

\[
x - \hat{x}_r \in \mathcal{N}(b_r, P_r)
\]

with \( b_r = (P_H - P_r) x \). The bias depends on the neglected singular values.

When \( x^T (U_n U_n^T) x + r < n \) there exists a minimum least-squares estimate with a reduced model. The mean squared estimate will even improve if \( r = 0 \) when the signal to noise ratio is \( \leq 1 \).
APPLICATION

Engine Layout

To assess the performance of the different estimators described previously, a model of the small SR-30 turbojet [1] is used. The basic engine features a one-stage radial compressor, a reverse flow combustor, a one-stage axial turbine and a converging nozzle. Additionally, the SR-30 owned by the Turbomachinery Group has been equipped with a load cell and a calibrated air intake so that the thrust and the air mass flow can be measured. Figure 2 depicts the engine installed on its bench.

The present study relies on data simulated with a non-linear, aero-thermodynamic model of the engine. This steady-state model has been tuned with real measurements collected at the test bench, as available in [13], thus providing a relatively accurate non-linear description of the behavior of the turbojet.

The selected sensor suite for use in the estimation process is reported in Table 1. As the engine is on a test cell, air mass flow and thrust are available to perform the estimation.

Monte-Carlo Simulations

Monte-Carlo simulations have been implemented to evaluate the estimators. The control parameters are the fuel flow and the engine inlet temperature and pressure while the performance parameters are the compressor and turbine efficiency and flow factors. Their respective reference values are given in Table 2. The three values of fuel flow correspond to spool speeds of respectively 55000 RPM, 65000 RPM and 75000 RPM.

The sampling procedure consists in simulating the engine with control and performance parameters subjected to an independent Gaussian distribution, the standard deviation of which is stated in Table 2, around their reference values. It should be noticed that the realization of the performance parameters is the same for the three operating points. On the contrary, the realization of the control parameters differs with the operating point. In the present application, this sampling procedure is repeated 1000 times for each operating point.

Then zero-mean, white Gaussian noise is added to the simulated measurements. The magnitude of this measurement noise is specified in Table 1.

To build the estimators, the engine model is linearized at the three operating points defined in Table 2. The estimation of the parameters is performed for different values of the prior parameter $\alpha$ (ranging from zero to unity) and of the order of the reduced model (ranging from full order down to order one) to study their influence on the estimation process. The order of the model is

| Table 1. ENGINE MEASUREMENTS ($y$ vector) |
|-----------------|-----------------|-------------|
| Label | Description | $\sigma$ | Units |
| $P3$ | compressor outlet pressure | 200 | [Pa] |
| $T3$ | compressor outlet temperature | 0.667 | [K] |
| $P5$ | turbine outlet pressure | 50 | [Pa] |
| $T5$ | turbine outlet temperature | 0.667 | [K] |
| $N$ | spool speed | 50 | [RPM] |
| $W1$ | inlet air mass flow | 1.33 | [g/s] |
| $FN$ | net thrust | 0.5 | [N] |

| Table 2. INPUT PARAMETER REFERENCE VALUE AND STANDARD DEVIATION FOR THE MONTE CARLO SIMULATIONS ($\theta$ vector) |
|-----------------|-----------------|-------------|
| Label | Reference value | $\sigma$ |
| $WFE_1$ | 2.55 g/s | 0.01 g/s |
| $WFE_2$ | 3.29 g/s | 0.01 g/s |
| $WFE_3$ | 4.51 g/s | 0.01 g/s |
| $T1$ | 288.15 K | 0.667 K |
| $P1$ | 101325 Pa | 33 Pa |
| SE2 | 1.0 | 0.1% |
| SWR2 | 1.0 | 0.3% |
| SE4 | 1.0 | 0.3% |
| SW4R | 1.0 | 0.2% |

Copyright © 2007 by ASME
here defined as the number of singular values kept in the decomposition (13). Notice that prior information is supposed to be available only for the control parameter deviations.

Finally, in order to process an equal amount of information, the single point estimator is run on three batches of data obtained around the 75000-RPM operating point.

**Singular Values of the System Matrix**

The sensitivity of the estimation problem can be assessed by analyzing the singular values of the model matrix $H$. Figure 3 presents the natural logarithm of the singular values of the system matrices $H_{SP}$ and $H_{MP}$. The single point problem encounters a large difference between the smallest and the largest singular values. Considering three operating regimes causes a better conditioning of the system matrix which is assessed by the ratio of the maximum to the minimum singular values. It can be seen that all singular values increase when considering the multi-point estimation.

![Figure 3. Singular values of the model matrix for single- and multi-point estimation (logarithmic scale)](image)

**Figure of Merit**

The root mean square of the estimation error, defined in equation (15), is selected as a figure of merit to compare the estimators. It gathers both the bias part and the variance part of the estimation error.

$$mse = \sqrt{\frac{1}{1000} \sum_{k=1}^{1000} (\hat{\theta}_{p,k} - \theta^*_p)^2}$$

(15)

where $\theta^*_p$ are the true values of the performance parameters for the $k$-th sample.

**Results**

Figure 4 depicts the natural logarithm of the mean square error ($mse$) as a function of the prior parameter $\alpha$ and the order of the model for the single point estimator.

![Figure 4. Performance of the single point estimator for various values of the prior parameter $\alpha$ and order of the model](image)

It can be seen that the classical least-square estimator, characterized here by an order of 13 (full order model) and a prior parameter of zero (no regularization) is the worst one. Large improvement in the estimation can be achieved by reducing the order model up to a certain point, as will be discussed below, or/and by introducing some prior knowledge about the deviations of the control parameters. More than six orders of magnitude can be gained in the $mse$ by selecting the most appropriate estimator.

The results obtained when using the multi-point estimation approach are represented in Figure 5 in the same way as for the single point estimation.

In this case, the classical least-square estimator is still the worst one, but achieves a lower $mse$ than in the case of single point estimation. Reducing the $mse$ can be achieved through regularization or/and reduced order estimation here also. The $mse$ can be reduced by about two orders of magnitude if the optimal estimator is chosen.

A more detailed analysis of the influence of the two tuners of the estimator will give guidelines for the choice of the most suited values for the regularization parameter and for the order of the model. The upper graph in Figure 6 shows the evolution of the $mse$ with respect to the order of the model for a fixed value of the prior parameter ($\alpha = 1.0$). Starting with the full order model (order = 13), the $mse$ first decreases and reaches a minimum for a model order of 11. As stated in the theory, this improvement
is due to a decrease in the variance of the estimated variables. However, it can be seen that the \( \text{mse} \) increases again if the model is further truncated. This effect is due to the contribution of the bias term on the identified parameters. Consequently, for the present application the optimal order of the reduced model is 11.

The lower part of Figure 6 depicts the evolution of the \( \text{mse} \) with respect to the amount of prior information that is included in the control parameter deviations for a fixed value of the order model (order = 11). The \( \text{mse} \) exhibits a monotone decrease as the contribution of the a priori knowledge increases. This is due to a refinement in the estimation of the control parameter deviations which in turn results in an improvement in the performance parameter estimation.

To conclude this analysis, it has to be pointed out that, for the present application, a multi-point approach of the estimation problem leads to slightly better results. Indeed, the \( \text{mse} \)'s for the single point (SP) and multi-point (MP) scheme are very close to each other. The explanation lies in the fact that the three operating points considered in this study (55000, 65000 and 75000 RPM) are too close, which can be assessed by computing the norm of the difference between the system matrices related to these power settings. Accordingly, the wealth of information available to the estimator is only slightly higher for the MP case than for the SP case.

**CONCLUSION**

The problem of parameter estimation applied to turbine engine has been reformulated in such a way that single- or multi-point, (non-)regularized and (non-)reduced order estimation fits in a unified framework. The paper extends previous work on singular value decomposition based model reduction. It shows how the reduction in estimator variance must be balanced against the bias on the estimate introduced by solving the inverse problem in a reduced order space.

A comparison of the different estimators has been made based on a model developed using test data collected for a turbojet engine. It has been shown that an optimal combination of a regularization parameter and the order of the model achieves the best estimator performance.

The results presented in this paper tend to show that a practical implementation of multi-point estimation hardly achieves the possible improvements demonstrated in the appendix. It should also be kept in mind that the present study did not take the inevitable presence of model error into account, which could further degrade the advantages of the multi-point estimator.

**ACKNOWLEDGEMENTS**

The authors wish to thank Mr. Quoc Khanh Mäi for the development of the simulation model of the SR-30 turbojet engine. The authors would also like to thank NFFP, The Swedish National Flight Research Program, for their financial support and especially Dr. Melker Härefors, manager of the project 435 at Volvo Aero Corporation in Trollhättan, Sweden.

**REFERENCES**

APPENDIX: Cramer-Rao bounds for single and multi-point performance estimation

There exists an argument which could favor linear multi-point estimation over single-point estimation if the operating point of the single point method is included in the multi-point one. The proof is based on carrying out two separate derivations, one for the single operating point case and one for the multiple operating point and is concluded by comparing the two bounds.

Single operating point

Assume the following structure for the single operating point model:

\[ y = \begin{pmatrix} H_{p,1} \\ H_{u,1} \end{pmatrix} \begin{pmatrix} \theta_p \\ \theta_u \end{pmatrix} H_{SP} \]

where \( \theta_p \) are performance parameters to be estimated, \( \theta_u \) are the control vector deviations. Form the Grammian of \( H_{SP} \):

\[ H_{SP}^T H_{SP} = \begin{pmatrix} H_{p,1}^T H_{p,1} \\ H_{u,1}^T H_{u,1} \end{pmatrix} \begin{pmatrix} H_{p,1} \\ H_{u,1} \end{pmatrix} \]

The Cramer-Rao lower bound is determined by analyzing the elements of the covariance matrix of \( H_{SP} \), i.e. the inverse of the Grammian, related to the performance parameters which writes:

\[
\begin{pmatrix} I \\ 0 \end{pmatrix} (H_{SP}^T H_{SP})^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix} = \left( H_{p,1}^T H_{p,1} - H_{p,1}^T H_{u,1} (H_{u,1}^T H_{u,1})^{-1} H_{u,1}^T H_{p,1} \right)^{-1} \\
= \left( H_{p,1}^T \left( I - H_{u,1} (H_{u,1}^T H_{u,1})^{-1} H_{u,1}^T \right) H_{p,1} \right)^{-1}
\]

where the matrix inversion lemma is used at the second step. Note that the matrix \( P_{u}^* \) can be interpreted geometrically as the symmetric orthogonal projection matrix.
Multiple operating point

Assume the following structure for the multi-point model:

\[
\begin{pmatrix}
    y_1 \\
    y_2
\end{pmatrix}
= \begin{pmatrix}
    H_{p,1} & H_{u,1} & 0 \\
    H_{p,2} & 0 & H_{u,2}
\end{pmatrix}
\begin{pmatrix}
    \theta_p \\
    \theta_{u,1} \\
    \theta_{u,2}
\end{pmatrix}
+ \begin{pmatrix}
    \epsilon_1 \\
    \epsilon_2
\end{pmatrix}
\tag{16}
\]

where \(\theta_p\) are the performance parameters, identical at the different operating points, \(\theta_{u,1}\) and \(\theta_{u,2}\) are the control vector deviations which depend on power setting. Form the Grammian:

\[
H_{MP}^T H_{MP} = \begin{pmatrix}
    H_{p,1}^T H_{p,1} + H_{p,2}^T H_{p,2} & H_{p,1}^T H_{u,1} & H_{p,2}^T H_{u,2} \\
    H_{u,1}^T H_{p,1} & H_{u,1}^T H_{u,1} & 0 \\
    H_{u,2}^T H_{p,2} & 0 & H_{u,2}^T H_{u,2}
\end{pmatrix}
\]

Again the elements of the covariance matrix related to the performance parameters are analyzed to obtain:

\[
(I \ 0 \ 0) (H_{MP}^T H_{MP})^{-1} \begin{pmatrix}
    1 \\
    0
\end{pmatrix}
= \begin{pmatrix}
    (H_{p,1}^T H_{p,1} - H_{p,1}^T H_{u,1} (H_{u,1}^T H_{u,1})^{-1} H_{u,1}^T H_{p,1})^{-1} \\
    H_{p,1}^T (I - H_{u,1} (H_{u,1}^T H_{u,1})^{-1} H_{u,1}^T) H_{p,1}
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
    (H_{p,2}^T H_{p,2} - H_{p,2}^T H_{u,2} (H_{u,2}^T H_{u,2})^{-1} H_{u,2}^T H_{p,2})^{-1} \\
    H_{p,2}^T (I - H_{u,2} (H_{u,2}^T H_{u,2})^{-1} H_{u,2}^T) H_{p,2}
\end{pmatrix}
\]

where the symmetric projection matrices \(P_{u,1}^\perp\) and \(P_{u,2}^\perp\) have been introduced. The relation is easily generalized to \(n\) operating points.

Comparison of the two bounds

The multi-point Cramer-Rao bound is lower than the single point bound if:

\[
(H_{p,1}^T P_{u,1}^\perp H_{p,1} + H_{p,2}^T P_{u,2}^\perp H_{p,2})^{-1} \leq (H_{p,1}^T P_{u,1}^\perp H_{p,1})^{-1}
\]

which is obviously the case as \(H_{p,i}^T P_{u,i}^\perp H_{p,i} > 0\) is true for any operating point.